S-Parameters and Power Gain Definitions

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1 S-Parameters Overview

1.1 The S-Parameters for a General 2 Port Network

Suppose we have a 2 port network whose S parameters we have measured accurately on a vector network analyzer (VNA) across some frequency range of interest. By a 2 port 'network' we mean almost any RF device that has 2 ports, with appropriate connectors, that we can safely measure on the VNA properly using the 'full 2-port' calibration method [1] [2] [3] [7]. We will assume that it produces accurate S-parameter values. We will also refer to it as the 'device under test' (DUT). It must contain linear components, for example resistors, inductors and capacitors and it must be operated under linear conditions. It can be an amplifier and contain non-linear devices such as transistors and diodes but, as a whole *the network must be operating linearly* [5] [13]. If it is an amplifier, it must be stable and operated with signal levels which allow it to work in the linear region of its transfer characteristic. It must not be operated at such a high signal level that it is into compression or at such a low signal level that might be below the background noise.

As we are considering calibration of the VNA to deliver full 2-port measurements, once this is done the instrument will alternately switch between applying the input power to one port, say port 1, and the other port, port 2. Although it is not a requirement, we will assume that the VNA has a nominal impedance of 50 Ω with coaxial (unbalanced) connectors. This probably the most standard in use today.

The linear requirement applies to the VNA as well. Most modern VNAs will have quite a good dynamic range but there is a risk of exceeding the maximum input power allowed. VNAs will generally have some means of adjusting the incident power level used for the measurement [5] [13]. Some thought is needed about how to set this before performing the measurement. For example, if the VNA has +10 dBm maximum input power, a good choice may be to adjust the maximum expected VNA input power to be around 10 dB less than this, 0 dBm. If we are measuring an amplifier we should have some estimate of its gain and output 1 dB compression point (P1dB). P1dB is, by definition, 1 dB into non-linearity so we need the signal to be comfortably below this value. If its P1dB is 0 dBm and we expect a gain of about 20 dB then, applying – 20 dBm at the amplifier input might be safe for the VNA but take the amplifier itself into compression so it would be safer to back off (reduce) the input by say another 10 dB to -30 dBm. Note that, on increasing the input power level, some amplifiers may quite abruptly go into compression.

Finally, we will assume that we have assigned the DUT ports correctly as '1' and '2' related to the VNA connections when the measurements were made. That should have been clear from the markings on the VNA. Normally port 1 on the VNA connects to what we define as port 1 on the DUT and similarly with port 2. There is no rule about which port on the VNA should be connected to which actual port on the DUT as long as we know which ports were in fact connected to correctly relate the measurements. Having said that, if we are measuring a non-reciprocal device such as an amplifier, for which power is intended to travel in one direction only, it is very common to make port 1 the input port and port 2 the output port. We should be comfortable with any sort of port definitions which might be defined.

We are not going to dwell too long on VNA setups but will look at the S-parameter results themselves and what can be done with them. Copious information on S-parameters and VNA measurements is available from VNA manufacturer's application notes and other references [1] [3] [4] [5] [7]. Now let us assume that we have a good set of S-parameters for the DUT, which we will simply refer to as a 2-port network, and let us connect it to an arbitrary source and load as shown schematically in Figure 1-1.

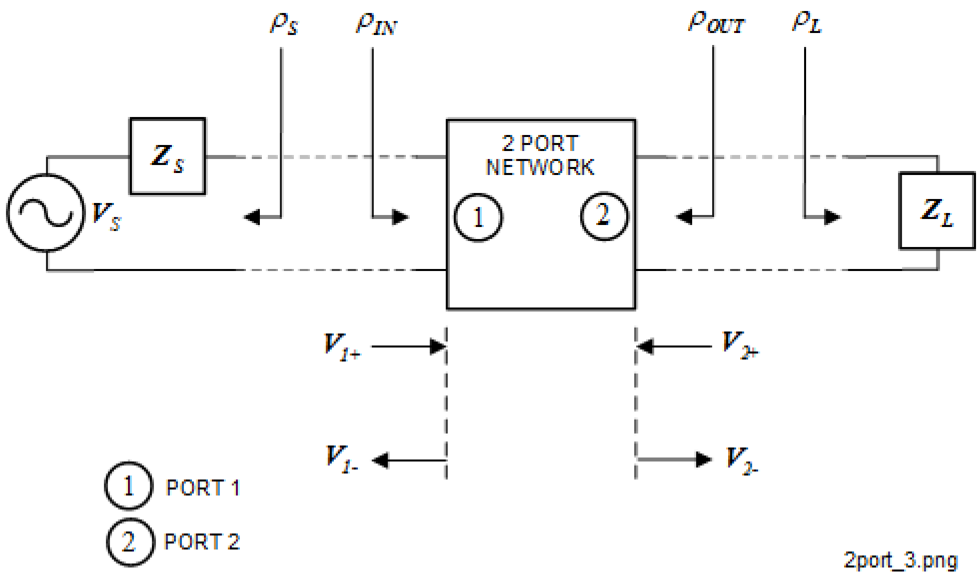

Figure 1-1 Some general 2 port network connected to an arbitrary source and load

1.2 Termination of a 2-Port Network in an Arbitrary Source and Load Impedance

Figure 1-1 shows the 2-port network connected to an arbitrary source on the left, which is represented by a Thevenin equivalent circuit, and an arbitrary load on the right [1] [2] [10] [11]. In this case there is no allowance for the delays inherent in any connecting cables. Therefore, at the highest operating frequency the wavelength within the connecting lines is assumed to be much greater than the physical dimensions of the components.

Using transmission line theory and the associated equations we must define a (nominal) system impedance or characteristic impedance (Z_0). This provides a reference impedance which is necessary for all transmission line systems. In a *perfectly matched* system using $Z_0 = 50\ \Omega$ the source and load impedances will of course be exactly 50 Ω [4] [8][14]. There would be no reflected waves and therefore no standing waves. However, in the real world (no pun intended) shown in Figure 1-1, the actual impedances will differ from 50 Ω, have reactive elements and be functions of other parameters, for example frequency and temperature. Therefore, standing waves will exist at ports 1 and 2.

We will generically refer to the 2-port network as *a network* but it could be an amplifier, attenuator, filter or anything else which operates linearly and for which we have a reliable set of S-parameter measurements. The ports have been named 1 and 2 as shown. In this case, we are considering port 1 to be the input and port 2 the output. It does not matter how we number the ports but we do need some sort of referencing system to ensure that we keep track of the correct port connections. The port numbers we actually choose will relate to the indices of the S-parameter matrix elements.

Unless stated otherwise, all values are complex quantities because they are all amplitude and phase sensitive. This applies to all of the electrical quantities: voltages, reflection coefficients, impedances, gains, losses. It may even be applied to power, but we are mostly only interested in the real part.

The wave directions can be confusing. Ultimately, in the connected system, we require the power to flow from the source on the left to the sink (load) on the right. Note that the wave directions either side of the network have positive subscripts into the associated port and negative subscripts out of the associated port. These are consistent with how the network was measured using the VNA as described in Section 1.1.

In the algebra which follows, we must keep track of which quantities are (linear) complex and which are just magnitudes or real values [2] [9] [10]. We will assume that all of the parameters have been measured at the same frequency.

With reference to Figure 1-1:

Z_S is the source impedance (Ω).

Z_L is the load impedance (Ω).

ρ_S is the reflection coefficient looking into the source only (unitless).

ρ_L is the reflection coefficient looking into the load only (unitless).

ρ_{IN} is the reflection coefficient looking into port 1 of the network whilst the load is connected and the source disconnected (unitless).

ρ_{OUT} is the reflection coefficient looking into port 2 of the network whilst the source is connected and the load disconnected (unitless).

The forward and reverse voltage waves listed below relate to the connected system and were chosen to be consistent with how the stand alone S-parameters were measured by the VNA: into the port (positive) and out of the port (negative).

V_{1+} is the forward voltage wave incident at port 1 (V).

V_{1-} is the reverse voltage wave reflected at port 1 (V).

V_{2+} is the forward voltage wave incident at port 2 (V).

V_{2-} is the reverse voltage wave reflected at port 2 (V).

The reflection coefficient at a port is defined as the *reflected* complex voltage divided by the *incident* complex voltage, so the reflection coefficient itself is also complex [2] [12]. Therefore, in the connected cascade shown in Figure 1-1, the following reflection coefficients are defined as follows:

$$\rho_{IN} = \frac{V_{1-}}{V_{1+}} \tag{1.1}$$

$$\rho_{OUT} = \frac{V_{2-}}{V_{2+}} \tag{1.2}$$

For ρ_S and ρ_L, noting that we have defined the incident and reflected voltages looking into the ports of the network, therefore:

$$\rho_S = \frac{V_{1+}}{V_{1-}} = \frac{1}{\rho_{IN}} \tag{1.3}$$

$$\rho_L = \frac{V_{2+}}{V_{2-}} = \frac{1}{\rho_{OUT}} \tag{1.4}$$

The convention for the forward and reflected voltages will be clear from the diagram. The subscript is associated with the port, 1 or 2 and includes a positive sign if the wave is entering the port or a negative sign if the wave is emerging from the port.

1.3 Power Waves and S-Parameter Definitions

For an n-port network, the scattering or S-parameter matrix is an n by n matrix. In our case the network has 2 ports so $n=2$ and we will denote the associated S-parameter matrix by the uppercase letter 'S' in bold italic, $\boldsymbol{S}$ [1] [2] [3] [7]. Regular type will be used for each of the elements within the matrix. The first and second subscripts of each element will be the row and column respectively. Therefore:

$$\boldsymbol{S}=\begin{pmatrix} \mathrm{S}_{11} & \mathrm{S}_{12} \\ \mathrm{S}_{21} & \mathrm{S}_{22} \end{pmatrix} \qquad (1.5)$$

With S-parameters it is useful to deal with quantities which are known as power waves [4]. These are still complex values and are related to the forward and reverse voltages at each of the ports, normalised by the square root of the system impedance which, by definition, is a real quantity. We use the lower case letter a to denote a power wave incident at a port and the lower case letter b for a power wave that is reflected from a port. In each case we include a subscript according to the port number. So, for example a_1 is the power wave incident at port 1 and b_2 is the power wave reflected from port 2. For the 2-port network, the power waves are related to the incident and reflected voltages at each of the ports as follows:

$$a_1=\frac{V_{1+}}{\sqrt{Z_0}} \qquad (1.6)$$

$$a_2=\frac{V_{2+}}{\sqrt{Z_0}} \qquad (1.7)$$

$$b_1=\frac{V_{1-}}{\sqrt{Z_0}} \qquad (1.8)$$

$$b_2=\frac{V_{2-}}{\sqrt{Z_0}} \qquad (1.9)$$

The following matrix defines the 2 port S-parameter matrix in terms of power waves [1] [3] [7].

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}=\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \qquad (1.10)$$

Multiplying out the matrices gives

$$b_1=S_{11}a_1+S_{12}a_2 \qquad (1.11)$$

and

$$b_2=S_{21}a_1+S_{22}a_2 \qquad (1.12)$$

(1.11) and (1.12) are used to define the individual S-parameter matrix elements. By definition, if port 2 was loaded with precisely the characteristic impedance (Z_0) there would be no reflected wave from the load:

- S_{11} is defined with $a_2=0$, so

$$S_{11} = \frac{b_1}{a_1} \tag{1.13}$$

- S_{12} is defined with $a_1 = 0$, so

$$S_{12} = \frac{b_1}{a_2} \tag{1.14}$$

- S_{21} is defined with $a_2 = 0$, so

$$S_{21} = \frac{b_2}{a_1} \tag{1.15}$$

- S_{22} is defined with $a_1 = 0$, so

$$S_{22} = \frac{b_2}{a_2} \tag{1.16}$$

As previously noted, the required conditions $a_1 = 0$ and $a_2 = 0$ may be achieved by terminating the associated port with a high quality load whose impedance is exactly Z_0. This is the principle used by a vector network analyzer (VNA) to measure S-parameters. High specification VNAs also include multi-port error correction algorithms, whose purpose is to correct for uncertainties of the source and load impedances themselves.

1.4 Complex Power and Power Gain Definitions

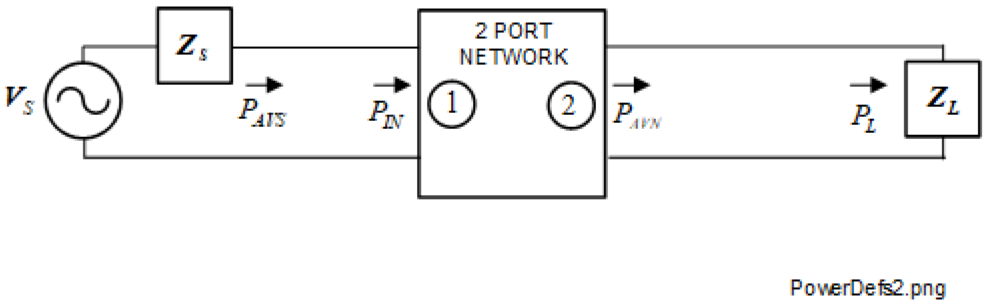

Figure 1-2 The 2-port network connections showing the power definitions

Figure 1-2 is a schematic for the same configuration as that shown in Figure 1-1, but in this case it shows the power flow definitions at each of the stages as follows [6]:

- Power available from the source, P_{AVS}.
- Input power to the network, P_{IN}.
- Power available from the network, P_{AVN}.
- Power dissipated in the load, P_L

These will be used for the power gain definitions.

To arrive at the power input to the network P_{IN} we need to find an expression for the average (or mean) power at the same point moving towards the load. Average power is the correct power definition applicable to the power gain definitions in this case, as we are using pure sinusoidal sources. There is no form of modulation, distortion or noise under consideration. Average power is calculated by its heating effect *averaged over many cycles*. If the power originates from sinusoidal voltage and current waveforms which are in phase, the resulting (instantaneous) power waveform will have a period that is half of that of either of the original waveforms, or twice the frequency. Its waveform will not be sinusoidal because of the square law relationship between the sinusoidal voltage (or current) and instantaneous power.

There are many other definitions of power, in particular *peak power* which is commonly specified in radar systems or modulated communications systems. The term root mean square (RMS) power or 'RMS power' is sometimes used ambiguously when actually referring to average power. This is generally understood to mean average power derived from RMS voltage and/or current waveforms.

1.4.1 Useful Complex Relationships

If A is any complex number, then

$$\mathrm{Re}(A) = \frac{1}{2}\left(A + A^*\right) \tag{1.17}$$

where $\mathrm{Re}(\ )$ means 'real part of' and the asterisk superscript means 'complex conjugate of'. So, for example, if $A = x + jy$, then $A^* = x - jy$.

If B is another complex number then for the product of the real parts of each of the complex numbers A and B is given by

$$\begin{aligned}\mathrm{Re}(A)\mathrm{Re}(B) &= \frac{1}{4}\left(A + A^*\right)\left(B + B^*\right)\\ &= \frac{1}{4}\left(A^*B + AB\right) + \frac{1}{4}\left(AB^* + A^*B^*\right)\\ &= \frac{1}{2}\mathrm{Re}\left(A^*B + AB\right)\end{aligned} \tag{1.18}$$

(1.18) is true because

$$\left(AB^* + AB\right)^* = A^*B + A^*B^* \tag{1.19}$$

It is understood that the magnitude of a complex number $|A|$ can be inferred from Pythagoras's theorem applied to the magnitudes of its real and imaginary components which, by definition, are at right angles, so that if $A = x + jy$, then

$$\begin{aligned}|A| &= \sqrt{x^2 + y^2}\\ |A|^2 &= x^2 + y^2\end{aligned} \tag{1.20}$$

Another useful relationship is that the magnitude squared of a complex number is the product of the complex number itself and its conjugate, so

$$AA^* = |A|^2 \tag{1.21}$$

This may be verified by sketching vectors for a complex number and its conjugate on an argand diagram. Both will have the same magnitude.

If we have two complex numbers A and B such that

$$A = x + jy \tag{1.22}$$

and

$$B = p + jq \tag{1.23}$$

Then some complex arithmetic will confirm the following relationships

$$\begin{aligned} |AB| &= |A||B| \\ \left|\frac{A}{B}\right| &= \frac{|A|}{|B|} \end{aligned} \tag{1.24}$$

Returning to Figure 1-1, we can use (1.18) to calculate the real power W at port 1, the input to the network, arising from the *total input voltage* V_1 (without any + or – sign in the subscript). The input total voltage is the voltage that results from the complex sum of the forward (or incident) voltage V_{1+} and the reflected voltage V_{1-}, so [1] [2] [3] [7]

$$\begin{aligned} W &= \frac{\left[\operatorname{Re}\left(V_1\right)\right]^2}{Z_0} \\ &= \frac{1}{2Z_0}\operatorname{Re}\left(V_1 V_1^* + V_1^2\right) \\ &= \frac{1}{2Z_0}\operatorname{Re}\left(|V_1|^2 + V_1^2\right) \end{aligned} \tag{1.25}$$

The expression for W contains the constant value $|V_1|^2$ which is the square of the amplitude of the total input voltage of the network. A sinusoidal (strictly co-sinusoidal) function of time in exponential form, V_1 could be expressed as

$$V_1 = \hat{V}_1 \cos \omega t \tag{1.26}$$

Using Euler's identity and complex exponential format for a general variable x of the form [9]

$$e^{jx} = \cos x + j \sin x \tag{1.27}$$

Then

$$\begin{aligned} V_1 &= \hat{V}_1 \cos \omega t \\ &= \operatorname{Re}\left(\hat{V}_1 \cos \omega t + j\hat{V}_1 \sin \omega t\right) \\ &= \operatorname{Re}\left(\hat{V}_1 e^{j\omega t}\right) \end{aligned} \tag{1.28}$$

where
V_1 is the *instantaneous* value of the sinusoidal voltage
$\hat{V}_1$ is the peak value (amplitude) of the voltage
$\omega = 2\pi f$ is the angular frequency of the source in radian per seconds (rad/s), and
f is the frequency in Hertz (Hz).

Following the convention in electrical engineering for complex exponentials that the real part operator may be omitted and the sinusoidal waveform may simply be expressed as

$$V_1 = \hat{V}_1 e^{j\omega t} \tag{1.29}$$

The amplitude of the sinusoidal voltage may alternatively be expressed as a magnitude in the form $|V_1|$, therefore

$$V_1 = |V_1| e^{j\omega t} \tag{1.30}$$

From (1.25) the term V_1^2 is given by squaring the expression in (1.30), that is

$$V_1^2 = |V_1|^2 e^{j2\omega t} \tag{1.31}$$

This represents a cosine wave at twice the fundamental frequency. Like any sine or cosine waveform, it will have a mean of zero over many cycles. For the average power W_{MEAN} therefore may be given by either of the following expressions.

$$\begin{aligned} W_{MEAN} &= \frac{1}{2Z_0}\operatorname{Re}\left(VV^*\right) = \frac{|V|^2}{2Z_0} \\ &= \frac{1}{2}Z_0\operatorname{Re}\left(II^*\right) = \frac{Z_0}{2}|I|^2 \\ &= \frac{1}{2}\operatorname{Re}\left(VI^*\right) = \frac{1}{2}|V||I| \end{aligned} \tag{1.32}$$

1.5 Bi-directional Power Along a Transmission Line

The configuration shown in Figure 1-2 is like an unmatched transmission line. There will be components of forward and reflected voltages and currents that will create corresponding (total or standing wave) voltages and currents. The following equations are based on the equations associated with transmission line theory [1] [2] [10] [11]. They describe the total instantaneous voltage, made up of the forward and reverse voltages; and the total instantaneous current, made up of the forward and reverse currents.

The total instantaneous voltage at port 1:

$$V_1 = V_{1+} + V_{1-} \tag{1.33}$$

The total instantaneous current at port 1:

$$\begin{aligned} Z_0 I_1 &= V_{1+} - V_{1-} \\ I_1 &= \frac{1}{Z_0}\left(V_{1+} - V_{1-}\right) \end{aligned} \tag{1.34}$$

The instantaneous power is the product $V_1 I_1$. This may be obtained by multiplying (1.33) and (1.34), then substituting expressions for V_{1+} and V_{1-} in terms of the forward and reverse power waves, a_1 and b_1 respectively, obtained from (1.6) and (1.8) as follows:

$$\begin{aligned} Z_0 V_1 I_1 &= \left(V_{1+} + V_{1-}\right)\left(V_{1+} - V_{1-}\right) \\ V_1 I_1 &= \frac{1}{Z_0}\left(V_{1+}{}^2 - V_{1-}{}^2\right) \\ &= \frac{V_{1+}{}^2}{Z_0}\left(1 - \frac{V_{1-}{}^2}{V_{1+}{}^2}\right) \\ &= \frac{V_{1+}{}^2}{Z_0}\left(1 - \rho_{IN}{}^2\right) \end{aligned} \qquad (1.35)$$

The last line of (1.35) used the definition of ρ_{IN} from (1.1). Remembering that the voltages and currents considered in (1.35) are instantaneous values of sinusoidal waveforms, so the product $V_1 I_1$ is also instantaneous. From the discussions in Section 1.4, this may be converted to a mean power P_{IN} by using the first line of (1.32), changing V_{1+}, V_{1-} and therefore ρ_{IN} to their magnitude values. This provides the following result.

$$P_{IN} = \frac{\left|V_{1+}\right|^2}{2Z_0}\left(1 - \left|\rho_{IN}\right|^2\right) \qquad (1.36)$$

1.6 Unilateral and Bilateral Properties of the Network

Looking at Figure 1-1 again, consider what might happen to ρ_{IN} if we changed Z_L (and therefore ρ_L).

If the network was a reasonably high gain amplifier the answer might be ‘not much’ or by some negligible amount. In some cases amplifiers are deliberately designed as ‘buffers’ precisely for this reason [5] [13]. However one of the important advantages of S-parameters is that *they take account of the effects of signals in both directions.* All such signals are expressed in both amplitude and phase. If we are using S-parameters to design a 2-port network, We do not want to get this wrong. Otherwise, if we are trying to design an amplifier, it may not amplify and instead oscillate. If we are trying to design an oscillator, it may not even start to oscillate when switched on. A perfect buffer amplifier is an example of a unilateral device (the signal transmission is in only one direction). More generally, most 2 port devices have at least some bilateral properties which must be taken into account.

Using the definitions of the a and b power waves from equations (1.6), (1.7), (1.8) and (1.9) and substituting them into (1.11) and (1.12)

$$V_{1-} = S_{11} V_{1+} + S_{12} V_{2+} \qquad (1.37)$$

From (1.4)

$$V_{2+} = \rho_L V_{2-} \qquad (1.38)$$

Substituting (1.38) into (1.37):

$$V_{1-} = S_{11}V_{1+} + S_{12}\rho_L V_{2-} \tag{1.39}$$

A similar substitution for V_{2-} results in

$$V_{2-} = S_{21}V_{1+} + S_{22}\rho_L V_{2-} \tag{1.40}$$

Re-arranging (1.39) in terms of V_{2-} gives

$$V_{2-} = \frac{V_{1-} - S_{11}V_{1+}}{S_{12}\rho_L} \tag{1.41}$$

Similarly, from (1.40)

$$V_{2-} = \frac{S_{21}V_{1+}}{1 - S_{22}\rho_L} \tag{1.42}$$

Equating (1.41) and (1.42), after some re-arrangement, yields

$$V_{1-} = V_{1+}\left(S_{11} + \frac{S_{12}S_{21}\rho_L}{1 - S_{22}\rho_L}\right) \tag{1.43}$$

Thus

$$\rho_{IN} = \frac{V_{1-}}{V_{1+}} = S_{11} + \frac{S_{12}S_{21}\rho_L}{1 - S_{22}\rho_L} \tag{1.44}$$

A similar argument applied to ρ_{OUT} gives the following result

$$\rho_{OUT} = \frac{V_{2-}}{V_{2+}} = S_{22} + \frac{S_{12}S_{21}\rho_S}{1 - S_{22}\rho_S} \tag{1.45}$$

Equations (1.44) and (1.45) are very useful in S-parameter design and matching problems [1] [2] [3] [7]. Equation (1.44) tells us how the reflection coefficient that we see at the input to the network ρ_{IN} is affected by the output loading of the network. Equation (1.45) describes how the output reflection coefficient ρ_{OUT} is affected by the source impedance.

Some 2-port devices are said to be symmetric if they will have identical values for S_{12} and S_{21}, together with $S_{11} = S_{22}$. For example, we would expect a perfect attenuator to have the same attenuation for whichever direction the power flows through it, so it would have symmetrical S-parameters.

The product $S_{12}S_{21}$, also known as the open loop gain, is an important quantity for many 2-port networks. If its input port was at port 1, S_{21} would be a linear measure of what would happen to the signal applied at the input and arriving at the output, port 2. S_{21} is known as the linear transmission, or linear gain in the case of an amplifier. After the input signal has been amplified (by S_{21}) then, in a real network with imperfections, a proportion of the signal might find its way from the output back to the input again if S_{12} was non-zero.

The open loop gain is important, if:

- we wish to design an amplifier that will not oscillate;
- we wish to actually design an oscillator.

In fact, transistors are available for oscillators which have designed-in finite $S_{12}S_{21}$ values. Transistors for use as amplifiers will be designed with the smallest possible value for $S_{12}S_{21}$. A 2-port network is said to be perfectly unilateral if $S_{12}S_{21}$ is zero. When we connect together several 2-port networks into a cascade, with the output of each network connected to the input of the next, we normally wish each network to be as unilateral as possible to avoid any risks of oscillations or other instabilities.

2 Power Gain Definitions

Refer again to Figure 1-1. We are now going to consider cases were the 2-port network under consideration is an amplifier, more specifically a power amplifier [6]. A power amplifier, as its name implies, is one that is designed to substantially increase the RF power of a signal applied at the input.

If we buy a power amplifier with *nominal* input and output impedances of, say 50 Ω, how sure can we be that it will be sufficiently close to 50 Ω for our requirements? We might expect it to differ slightly from 50 Ω but by how much and what can we tolerate happening to the phase? If the input impedance of a *unilateral* amplifier was $48 + j10\,\Omega$ for example, it will give quite a respectable return loss (about 20 dB relative to 50 Ω), in fact identical to if the input impedance had been $48 - j10\,\Omega$. If the impedance of the source was $48 - j10\,\Omega$ and input impedance of the amplifier was $48 + j10\,\Omega$, that would be a perfect conjugate match. Power would transfer perfectly from the source to the amplifier without any loss. Now suppose that the impedances of the source and amplifier happened to be identical at $48 + j10\,\Omega$. Because that is not a conjugate match, quite a lot of the power from the source would be reflected. In fact the magnitude of the reflection coefficient at the input, *but relative to the source impedance and not 50 Ω*, would only be 0.208, or a return loss of about 14 dB. That means that the reflected power at the amplifier input would be 14 dB less than the incident power, quite a significant waste and potential for instability, especially if we want to transfer high power.

It is for these type of situations that there are different definitions of power gain. Also you will remember that I conveniently assumed the amplifier to be perfectly unilateral as that approximation saves us lots of trouble. From this point onwards we will assume that the network is bilateral. That is to say that we will allow for signals in both directions so different loadings of the output can affect what is seen at the input, and vice versa.

Here again are the types of power gain that we first met in Section 1.4:

- Power available from the source, P_{AVS}.
- Input power to the network, P_{IN}.
- Power available from the network, P_{AVN}.
- Power dissipated in the load, P_L

There are 3 ways of defining power gain with the following (scalar) quantities:

- Operating Power Gain ($G_{OP} = P_L / P_{IN}$).
- Available Power Gain ($G_A = P_{AVN} / P_{AVS}$).
- Transducer Power Gain ($G_T = P_L / P_{AVS}$).

These are described in the following sections.

2.1 Operating Power Gain

The operating power gain G_{OP} is the ratio of the power dissipated in the load P_L to the power delivered to the input of the 2-port network P_{IN}, so

$$G_{OP} = \frac{P_L}{P_{IN}} \qquad (2.1)$$

From Figure 1-1, if the total voltage at the input: the standing wave which is made up of the forward and reverse waves, V_{1+} and V_{1-} respectively, is V_1, then

$$V_1 = V_{1+} + V_{1-} \tag{2.2}$$

Using the definition of input reflection coefficient in (1.1),

$$\begin{aligned} V_1 &= V_{1+} + V_{1-} \\ &= V_{1+} + \rho_{IN} V_{1+} \\ &= V_{1+}\left(1+\rho_{IN}\right) \end{aligned} \tag{2.3}$$

An alternative way of describing V_1 is in terms of the potential divider action of Z_S and Z_{IN} on the voltage at the source V_S , so

$$V_1 = V_S\left(\frac{Z_{IN}}{Z_S + Z_{IN}}\right) \tag{2.4}$$

Equating (2.3) and (2.4)

$$\begin{aligned} V_S\left(\frac{Z_{IN}}{Z_S + Z_{IN}}\right) &= V_{1+}\left(1+\rho_{IN}\right) \\ V_{1+} &= \left(\frac{V_S}{1+\rho_{IN}}\right)\left(\frac{Z_{IN}}{Z_S + Z_{IN}}\right) \end{aligned} \tag{2.5}$$

At this point it is useful to get expressions for the source and load impedances, Z_S and Z_L related to their associated reflection coefficients ρ_S and ρ_L respectively. When we express a reflection coefficient in terms of impedance we need to relate it to a reference impedance and without doubt the most useful would be the same system impedance that we used for the S-parameters, Z_0. Therefore, for the load,

$$\rho_L = \frac{Z_L - Z_0}{Z_L + Z_0} \tag{2.6}$$

Re-arranging in terms of the load impedance

$$Z_L = Z_0\left(\frac{1+\rho_L}{1-\rho_L}\right) \tag{2.7}$$

Similarly, for the source and input impedances

$$Z_S = Z_0\left(\frac{1+\rho_S}{1-\rho_S}\right) \tag{2.8}$$

and

$$Z_{IN} = Z_0\left(\frac{1+\rho_{IN}}{1-\rho_{IN}}\right) \tag{2.9}$$

There are of course similar expressions to (2.6) and (2.7) for all of the other reflection coefficients.

By substitution from (2.8) and (2.9) for Z_S and Z_{IN} into (2.5) after a little perseverance we will get the following result

$$V_{1+} = \frac{V_S}{2}\frac{\left(1-\rho_S\right)}{\left(1-\rho_S\rho_{IN}\right)} \tag{2.10}$$

Next we call on our study of average power in Section 1.4 and use (1.36) as follows:

$$\begin{aligned} P_{IN} &= \frac{1}{2Z_0}\left|V_{1+}\right|^2\left(1-\left|\rho_{IN}\right|^2\right) \\ &= \frac{\left|V_S\right|^2}{8Z_0}\frac{\left|1-\rho_S\right|^2}{\left|1-\rho_S\rho_{IN}\right|^2}\left(1-\left|\rho_{IN}\right|^2\right) \end{aligned} \tag{2.11}$$

Noting that V_{2-}, the reflected wave from port 2 is identical to the incident wave at the load, then the power delivered to the load P_L is, by the same reasoning,

$$P_L = \frac{\left|V_{2-}\right|^2}{2Z_0}\left(1-\left|\rho_L\right|^2\right) \tag{2.12}$$

A few more stages of quite tedious algebra are still required. From (1.40), in terms of V_{2-}

$$V_{2-} = \frac{S_{21}V_{1+}}{1-S_{22}\rho_L} \tag{2.13}$$

As the complex numbers are in the form of products and/or quotients, (2.13) can be written in magnitude form as follows

$$\left|V_{2-}\right| = \frac{\left|S_{21}\right|\left|V_{1+}\right|}{\left|1-S_{22}\rho_L\right|} \tag{2.14}$$

Substituting for $\left|V_{2-}\right|$ from (2.13) into (2.12)

$$P_L = \frac{\left|S_{21}\right|^2\left|V_{1+}\right|^2\left(1-\left|\rho_L\right|^2\right)}{2Z_0\left|1-S_{22}\rho_L\right|^2} \tag{2.15}$$

Substituting for $\left|V_{1+}\right|$ from (2.10) into (2.15)

$$P_L = \frac{\left|S_{21}\right|^2\left(1-\left|\rho_L\right|^2\right)\left|V_S\right|^2\left|1-\rho_S\right|^2}{8Z_0\left|1-S_{22}\rho_L\right|^2\left|1-\rho_S\rho_{IN}\right|^2} \tag{2.16}$$

Substituting for P_L from (2.16) and for P_{IN} from (2.11), the operating power gain G_{OP} is

$$G_{OP} = \frac{P_L}{P_{IN}} = \frac{|S_{21}|^2 \left(1 - |\rho_L|^2\right)}{\left(1 - |\rho_{IN}|^2\right) |1 - S_{22}\rho_L|^2} \tag{2.17}$$

As this is a power gain it is a ratio of powers, more accurately mean powers, for which phase is not applicable. It is clear from (2.17) that, although they were derived from complex quantities, the coefficients involved are actually all magnitudes so the result G_{OP} is a scalar quantity.

2.2 Available Power Gain

The available power gain G_A is the ratio of the power available from the 2-port network P_{AVN} to the power available from the source P_{AVS}, so

$$G_A = \frac{P_{AVN}}{P_{AVS}} \tag{2.18}$$

The (maximum) power available from the source P_{AVS} is when it is conjugatively matched to the input of the network. That is, when the input impedance of the network is the conjugate of the source impedance, or

$$\rho_{IN} = \rho_S^{\ *} \tag{2.19}$$

Substituting for ρ_{IN} from (2.19) into (2.11), and remembering that

$$\left|\rho_S^{\ *}\right|^2 = |\rho_S|^2 \tag{2.20}$$

then

$$\begin{aligned} P_{AVS} &= \frac{|V_S|^2}{8Z_0} \frac{|1 - \rho_S|^2}{\left|1 - |\rho_S|^2\right|^2} \left(1 - \left|\rho_S^{\ *}\right|^2\right) \\ &= \frac{|V_S|^2}{8Z_0} \frac{|1 - \rho_S|^2}{\left(1 - |\rho_S|^2\right)} \end{aligned} \tag{2.21}$$

Similarly to the source, the power available from the network is equivalent to the maximum power which can be delivered to the load. That is, when ρ_{OUT} is a conjugate match to the load reflection coefficient ρ_L, or

$$\rho_L = \rho_{OUT}^{\ *} \tag{2.22}$$

Therefore,

$$|\rho_L| = |\rho_{OUT}| \tag{2.23}$$

Substituting from (2.22) and (2.23) into (2.16)

$$P_{AVN} = \frac{|V_S|^2}{8Z_0} \frac{|S_{21}|^2 \left(1-|\rho_{OUT}|^2\right)|1-\rho_S|^2}{\left|1-S_{22}\rho_{OUT}{}^*\right|^2 |1-\rho_S\rho_{IN}|^2} \tag{2.24}$$

The following step is derived from (1.44)

$$|1-\rho_S\rho_{IN}|^2 = \frac{|1-S_{11}\rho_S|^2 \left(1-|\rho_{OUT}|^2\right)^2}{\left|1-S_{22}\rho_{OUT}{}^*\right|^2} \tag{2.25}$$

Substituting (2.25) into (2.24) gives the result

$$P_{AVN} = \frac{|V_S|^2}{8Z_0} \frac{|S_{21}|^2 |1-\rho_S|^2}{|1-S_{11}\rho_S|^2 \left(1-|\rho_{OUT}|^2\right)} \tag{2.26}$$

By substituting (2.21) and (2.26) into (2.18) the result for the available power gain G_A is

$$G_A = \frac{P_{AVN}}{P_{AVS}} = \frac{|S_{21}|^2 \left(1-|\rho_S|^2\right)}{|1-S_{11}\rho_S|^2 \left(1-|\rho_{OUT}|^2\right)} \tag{2.27}$$

2.3 Transducer Power Gain

The transducer gain G_T is the ratio of the power delivered to the load P_L to the power available from the source P_{AVS} , therefore

$$G_T = \frac{P_L}{P_{AVS}} \tag{2.28}$$

We already have expressions for P_L and P_{AVS} from (2.16) and (2.21) respectively, so the transducer power gain G_T is

$$G_T = \frac{P_L}{P_{AVS}} = \frac{|S_{21}|^2 \left(1-|\rho_S|^2\right)\left(1-|\rho_L|^2\right)}{|1-\rho_S\rho_{IN}|^2 |1-S_{22}\rho_L|^2} \tag{2.29}$$

2.4 Comparing the Power Gain Definitions

We have at last derived the expressions for operating power gain G_{OP} (2.17), available power gain G_{AV} (2.27) and transducer power gain G_T (2.29). These are repeated below for convenience

$$G_{OP} = \frac{P_L}{P_{IN}} = \frac{|S_{21}|^2 \left(1-|\rho_L|^2\right)}{\left(1-|\rho_{IN}|^2\right)|1-S_{22}\rho_L|^2} \tag{2.30}$$

$$G_A = \frac{P_{AVN}}{P_{AVS}} = \frac{|S_{21}|^2 \left(1-|\rho_S|^2\right)}{|1-S_{11}\rho_S|^2 \left(1-|\rho_{OUT}|^2\right)} \qquad (2.31)$$

$$G_T = \frac{P_L}{P_{AVS}} = \frac{|S_{21}|^2 \left(1-|\rho_S|^2\right)\left(1-|\rho_L|^2\right)}{|1-\rho_S \rho_{IN}|^2 |1-S_{22}\rho_L|^2} \qquad (2.32)$$

All three power gain definitions are of course scalar quantities. If we had a hypothetical system in which all the impedances were exactly equal to Z_0 then, as Z_0 is the system impedance, all of the magnitudes of the reflection coefficients would also be zero. This is implied by (2.6) in the case of ρ_L and similar other equations for the other reflection coefficients. Therefore, from (2.30), (2.31) and (2.32), for a perfectly matched system,

$$G_{OP} = G_{AV} = G_T = |S_{21}|^2 \qquad (2.33)$$

More realistically, the network might be an active device such as a transistor. Provided it is operating linearly and is unconditionally stable at the frequency under consideration, then our discussions are reliable. Today's technology cannot yet produce transistor architectures which have impedances near our most common system impedance of $50\,\Omega$ and maintain them over frequency, so various forms of matching are commonplace.

2.5 Power Gains in Practical Systems

Often in communications systems we are faced with needing to insert a *fairly arbitrary 2-port network* into a cascade of 2-port networks as shown in Figure 2-1. Figure 2-1 (a) shows that, at the beginning of the cascade, there would be some form of source, in this case represented by a Thevenin equivalent circuit with voltage source V_A and a source impedance Z_A. At the output of the cascade there would be a load, in this case Z_B. In general, Z_A and Z_B are both complex and frequency-dependent. The intermediate stages comprising the cascade may have many different, frequency dependent, input and output impedances. These may be considered equivalent to a single Thevenin equivalent circuit with source impedance Z_S at the input and a load of Z_L at the output. Now suppose we wish to insert a 2-port network S_K into the cascade, whose S-parameters are already known, as shown in Figure 2-1(b). If it was inserted directly into the cascade, *without any matching*, it would probably not be matched by chance so we could expect some reflected power at either port of the network, probably both.

In this case, if we were interested in how the network affects the overall power gain of the cascade, we would use the definition of transducer power gain G_T for the network, as defined in (2.32). By using (2.32) we would automatically take into account the mismatches present at the input and output of S_K. The impedance Z_S results in the reflection coefficient ρ_S and similarly for Z_L and ρ_L.

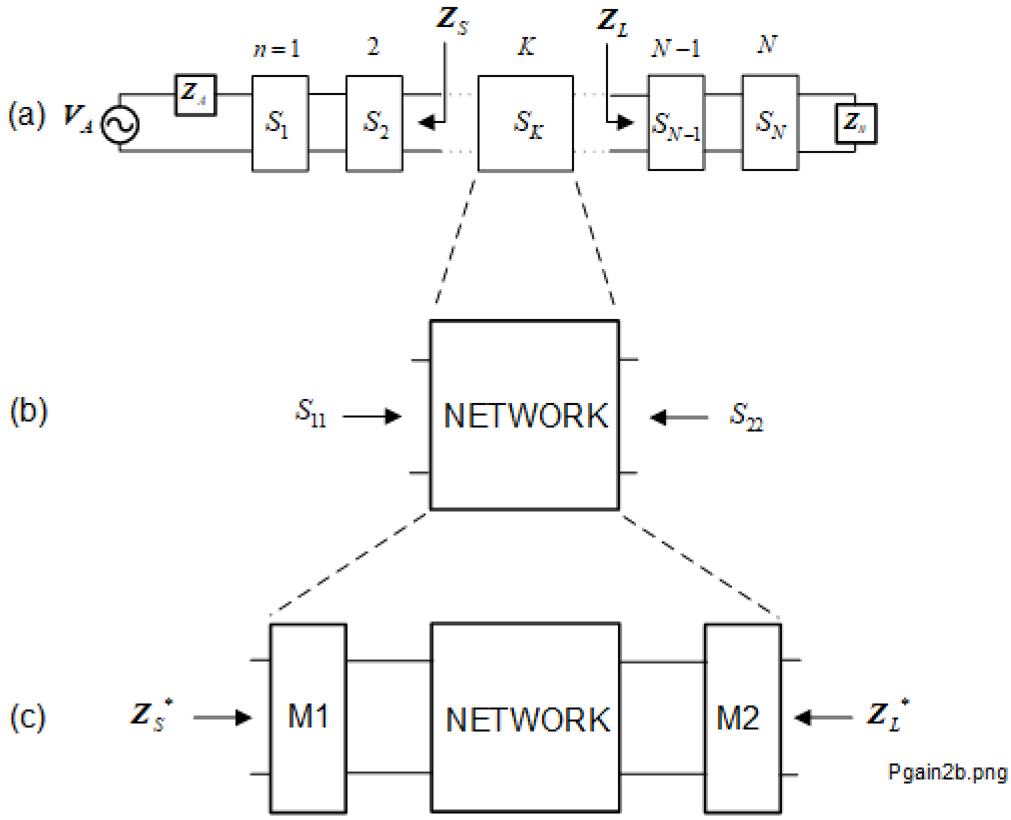

Figure 2-1 A schematic showing two ways of substituting a 2-port network into a cascade: unmatched (a) and simultaneously conjugately matched (b)

2.5.1 Matching for Maximum Power Transfer

By inserting the network S_K into the cascade as we did in Section 2.5, we did not make the assumption that S_K was unilateral. In other words, we are not assuming that we can simply match the input with a source impedance of S_{11}^* and the output with a load impedance of S_{22}^*. The S-parameters of the network were measured stand-alone using a system impedance Z_0 which is not necessarily anything like those we are trying to match to. In a network such as this which may not be unilateral, when it is inserted into the cascade, the impedance seen at the input may be affected by the output loading. Similarly the impedance seen at the output may be affected by the input or source loading. We would need to use (1.44) or (1.45) to determine the actual input impedance or output impedance.

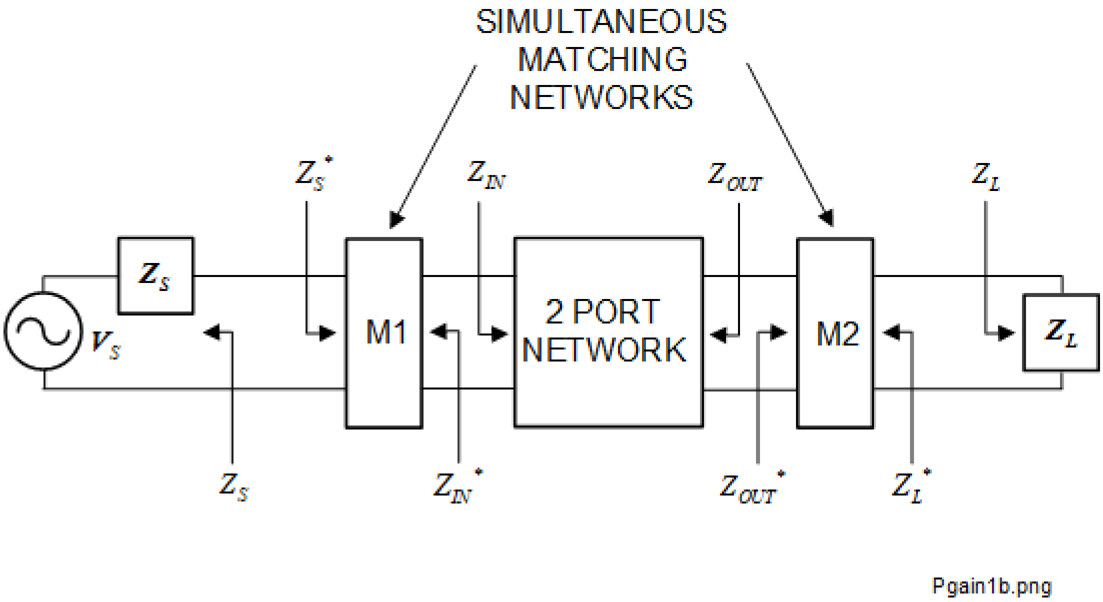

Figure 2-2 A generic 2-port network that is simultaneously conjugately matched to a source and load

To maximise the power transmitted along the cascade, one solution would be to design simultaneous input and output matching for the network S_K [8]. This is shown in Figure 2-1(c) and with more detail in Figure 2-2. M1 and M2 are matching networks positioned respectively at the input and output of the network S_K and they are designed to ensure that each input and output impedance seen is matched by a conjugate match in the opposite direction. The impedances and their conjugates are shown in Figure 2-2. In practice there are several prerequisites for matching of this type to work successfully, the most challenging requiring M1 and M2 to be sufficiently low loss across the intended operating frequency range.

Once successfully designed this form of matching will ensure that power is efficiently transferred from input to output and will be equivalent to the result in (2.33).

3 References

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